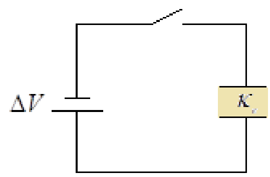
Homework 7 (Solutions): Capacitors

Problem 1. Hey look at that capacitor down there. It's one of those parallel plate-types (A = 50cm^2, d = 1mm, κ_e = 500). And the battery has a potential difference of 120V. Now say I flip the switch…



(a) What's the capacitor's capacitance?
Well,

$$C = \kappa_e \frac{A\varepsilon_0}{d} = (500)\frac{(50\times 0.01^2)(8.85\times 10^{-12})}{0.001} = 22\text{nF}$$

(b) What's the charge on the capacitor?
So the charge stored is:

$$Q = C\Delta V = (22\text{nF})(120\text{V}) = 2.64\mu\text{C}$$

(c) What's the energy stored?
Energy is:

$$PE = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}(22\text{nF})(120V)^2 = 160\mu\text{J}$$

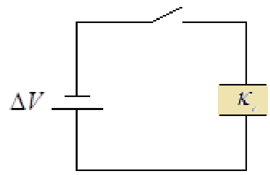
(d) What's the electric field within the capacitor?
Electric field is:

$$E = \frac{dV}{ds} = \frac{120\text{V}}{0.001\text{m}} = 120\,\text{kN/C}$$

(e) What's the force between the plates?
Force is:

$$F = qE = (2640\,\text{nC})(120\,\text{kN/C}) = 320\text{mN}$$

Problem 1´. Now say I keep the capacitor connected to the battery while I replace the κ_e = 500 dielectric with a κ_e = 1000 dielectric.



(a) What would be the new capacitance?
The capacitance would change. New capacitance is double the old one, since κ_e doubled.

$$C = 2(22\text{nF}) = 44\text{nF}$$

(b) Immediately after insertion, what would be the voltage across the capacitor?
The dielectric would immediately reduce the field by a factor of 2, since κ_e was doubled, and $E = E_0/\kappa_e$. And therefore, it would reduce the potential difference by the same factor. So the new ΔV would be:

$$\Delta V = \frac{120\,\text{V}}{2} = 60\,\text{V}$$

(c) A 'long' time after insertion what would be the potential difference across the capacitor?
The battery would charge the capacitor back up to 120V.

(d) What would be the new charge on the capacitor?
Charge would double, since C doubled, and ΔV ultimately remained the same.

$$Q = 2(2.64\mu\text{C}) = 5.28\mu\text{C}$$

(e) What would be the new potential energy?
Potential energy will double too, since C doubled and ΔV remained same.

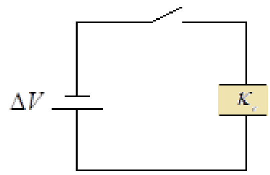
$$PE = 2(160\mu\text{J}) = 320\mu\text{J}$$

(f) What would be the new electric field?
The electric field remains the same though, since ΔV and d are not different.

(g) What would be the new force?
But since charge doubled, and E remained the same, the force will double:

$$F = 2(320\text{mN}) = 640\text{mN}$$

Problem 1´´. Now suppose that instead of inserting the new dielectric, while the capacitor was connected to the battery, I had disconnected the capacitor first, and then inserted the new $\kappa_e = 1000$ dielectric.



(a) What would the new capacitance be?
Well, C would double, since κ_e doubled.

$$C = 2(22\text{nF}) = 44\text{nF}$$

(b) What would be the new charge?
Q wouldn't change, since it has nowhere to go. So,

$$Q = 2.64\mu\text{C}$$

(c) What would be the new potential difference across the plates?
This would be reduced from the original by a factor of two, since the dielectric will reduce the field by that factor. So we'll have:

$$\Delta V = \frac{120\text{V}}{2} = 60\text{V}$$

(c) What would be the new potential energy?
Since C goes up by a factor of 2, and ΔV goes down by the same factor, and PE = (1/2)CΔV^2, the PE will go down by a factor of 2.

$$PE = \frac{160\mu\text{J}}{2} = 80\,\mu\text{J}$$

(d) What's the new electric field within the capacitor?
Field would be ΔV/d, which is:

$$E = \frac{\Delta V}{d} = \frac{60\text{V}}{0.001\text{m}} = 60\,\text{kN/C}$$

$$E = 120\,\text{kN/C}$$

(e) What's the new force on the plates?
New force is:

$$F = qE = (2.64\mu\text{C})(60\,\text{kN/C}) = 160\text{mN}$$

Problem 2. Before the advent of advanced dielectric materials, the only way to store a lot of charge was to have a 'big' capacitor (or a whooooole bunch of small ones). Say we wanted to store 1C of charge on a parallel plate capacitor with side lengths L, plate separation d, and air in between. We'd like to keep L as small as possible.

(a) Should we increase or decrease d, to keep L as small as possible.
Decreasing d would increase capacitance, and therefore allow us to reduce L.

(b) Decreasing d, inceases E, and brings us closer to the dielectric breakdown strength of the air. If we reduce d to the point where the field between the capacitor plates is the dielectric breakdown strength of air (E = 3MN/C), what would L have to be to store 1C of charge?
Well,

$$Q = C\Delta V = \kappa_e \frac{A\varepsilon_0}{d}\Delta V = \kappa_e A\varepsilon_0 \frac{\Delta V}{d} = A\kappa_e\varepsilon_0 E$$

So,

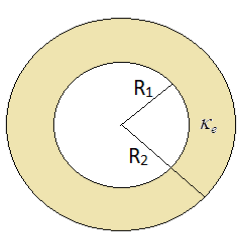
$$A = \frac{Q}{\kappa_e\varepsilon_0 E}$$

$$L = \sqrt{\frac{Q}{\kappa_e\varepsilon_0 E}} = \sqrt{\frac{1}{(1)(8.85\times10^{-12})(3\times10^{6})}} = 194\,\mathrm{m}$$

(c) Now suppose we ditch the air for a 'super' dielectric $\kappa_e = 2\times10^6$, with a breakdown strength of 12MN/C. What would L have to be to store 1C of charge?
Can use the same formula above. New L would be:

$$L = \sqrt{\frac{Q}{\kappa_e\varepsilon_0 E}} = \sqrt{\frac{1}{(2\times10^{6})(8.85\times10^{-12})(12\times10^{6})}} = 6.9\,\mathrm{cm}$$

Problem 3. In class, we derived an expression for the capacitance of a parallel plate capacitor. (a) Derive the formula for a spherical capacitor. Take it to have inner radius R_1, outer radius R_2, and filled in between with dielectric κ_e. Should get $4\pi\kappa_e\varepsilon_0 R_1 R_2/(R_2 - R_1)$.



So we start with:

$$
\begin{aligned}
Q &= C\Delta V \\
&= C\cdot\int_{R_1}^{R_2} \mathbf{E}\cdot d\mathbf{r} \qquad \text{it's just absolute value of } \Delta\text{V that matters in C formula} \\
&= C\cdot\int_{R_1}^{R_2} \frac{\mathbf{E}_0}{\kappa_e}\cdot d\mathbf{r} \\
&= C\cdot\int_{R_1}^{R_2} \frac{1}{\kappa_e}\frac{kQ}{r^2}dr \\
&= C\cdot\frac{kQ}{\kappa_e}\left(-\frac{1}{R_2}+\frac{1}{R_1}\right)
\end{aligned}
$$

And so we get:

$$C = \frac{\kappa_e}{k\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} = \frac{4\pi\kappa_e\varepsilon_0}{\frac{R_2-R_1}{R_1R_2}} = 4\pi\kappa_e\varepsilon_0\frac{R_1R_2}{R_2-R_1}$$

(b) Show that if we keep the distance between the plates, $R_2 - R_1$ constant, but make the radii very large, that this formula reduces to the parallel plate formula: $C = \kappa_e A\varepsilon_0/d$. This is why the parallel plate capacitor arrangement is more applicable than one might think.
So in the limit that the radii get large, but their separation is constant, then $R_1 \approx R_2$ = 'R'. Then $4\pi R_1R_2 \approx 4\pi R^2$, which is the area of the plates. And also, $R_2 - R_1$ is by definition, d. So we have:

$$C = 4\pi\kappa_e\varepsilon_0\frac{R^2}{R_2-R_1} \approx \kappa_e\frac{4\pi R^2\varepsilon_0}{R_2-R_1} = \kappa_e\frac{A\varepsilon_0}{d}$$

Problem 4. So….we defined the equivalent capacitor to be the one which stores the same change at the same voltage as the network of capacitors it replaces. So here's a question, does the equivalent capacitor also store the same *energy* as the capacitor network it replaces. Prove that it does, or doesn't, for the parallel and series combinations. You'll find the formulas $PE = (1/2)C\Delta V^2 = Q^2/2C$ to be useful.

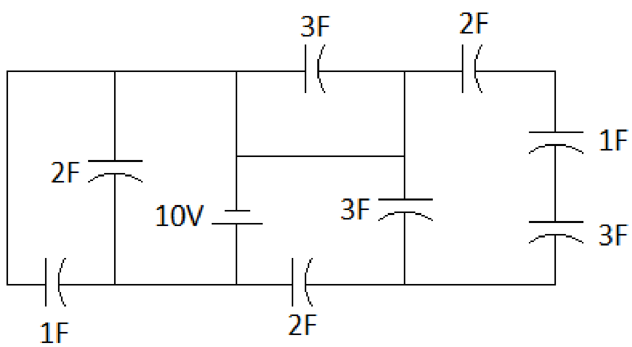
For parallel,

$$
\begin{aligned}
PE &= \frac{1}{2}C_{parallel}\Delta V_{parallel}^2 \\
&= \frac{1}{2}(C_1+C_2+\ldots+C_n)\Delta V^2 \\
&= \frac{1}{2}C_1\Delta V^2+\frac{1}{2}C_2\Delta V^2+\ldots+\frac{1}{2}C_n\Delta V^2 \\
&= PE_1+PE_2+\ldots.+PE_n
\end{aligned}
$$

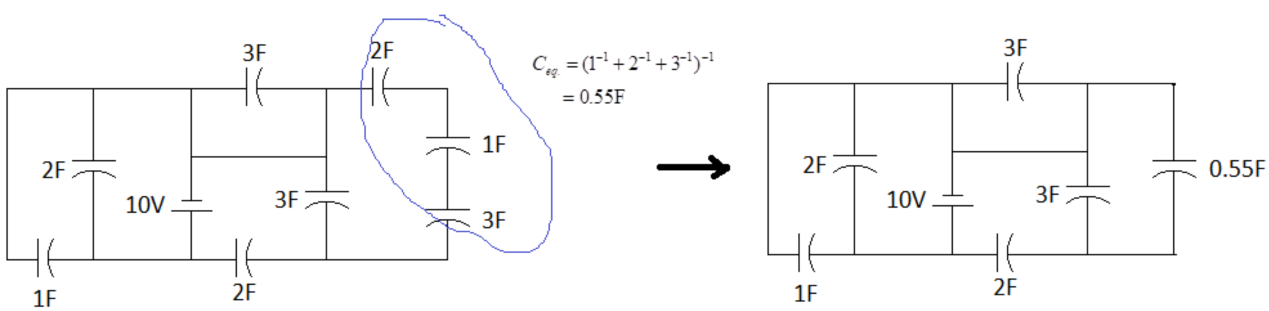
For series,

$$
\begin{aligned}
PE &= \frac{1}{2}\frac{Q_{series}^2}{C_{series}} \\
&= \frac{1}{2}\frac{Q^2}{C_{series}} \\
&= \frac{1}{2}Q^2\left(\frac{1}{C_1}+\frac{1}{C_2}+....+\frac{1}{C_n}\right) \\
&= \frac{1}{2}\frac{Q^2}{C_1}+\frac{1}{2}\frac{Q^2}{C_2}+...+\frac{1}{2}\frac{Q^2}{C_n} \\
&= PE_1 + PE_2 + ... + PE_n
\end{aligned}
$$

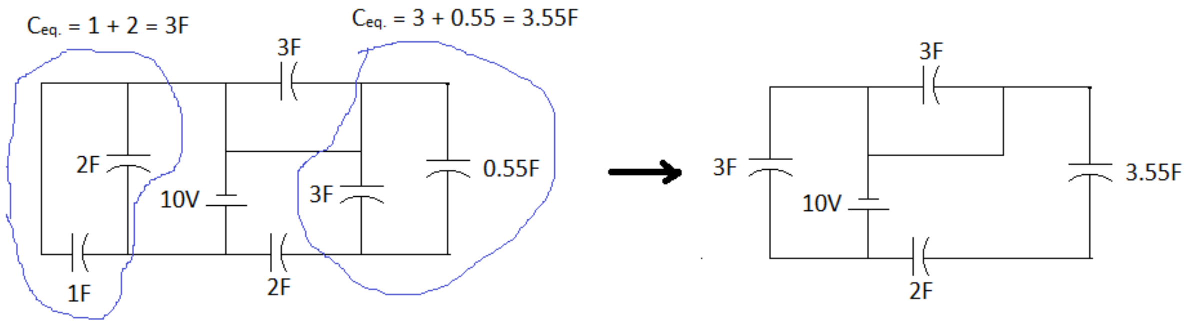
Problem 5. Consider the following network of capacitors. (a) Determine the charge on each one.



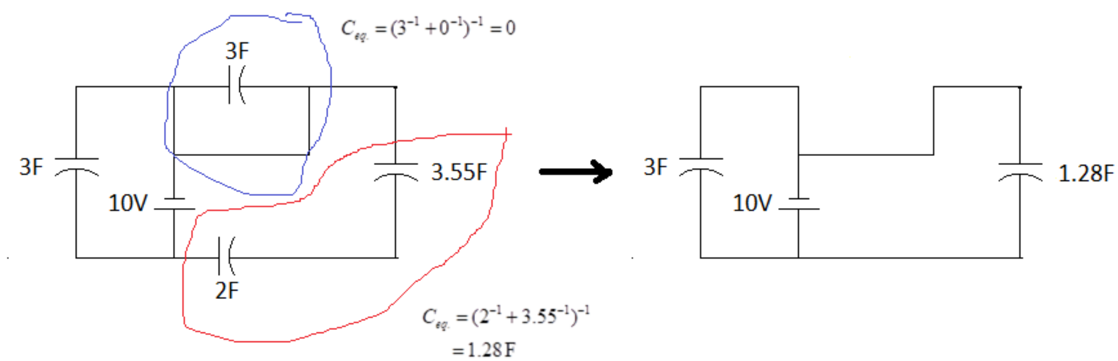
Reducing everything step by step,



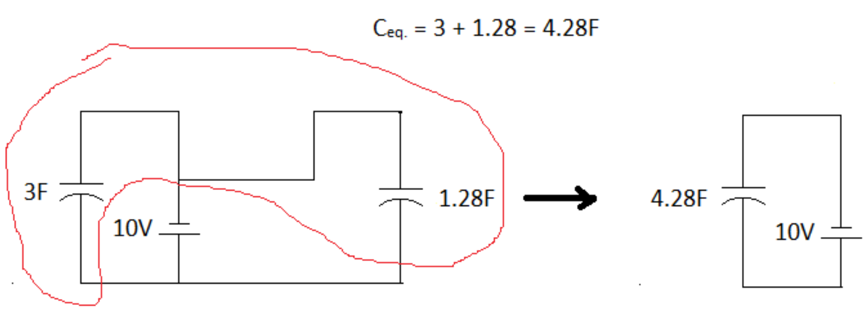
And,



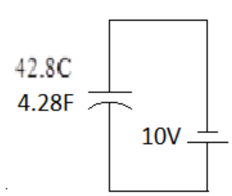
And, next…



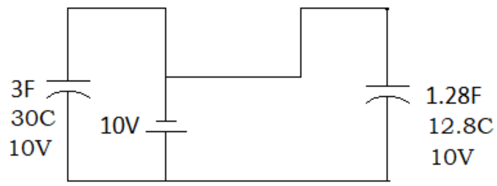
Neeeext,



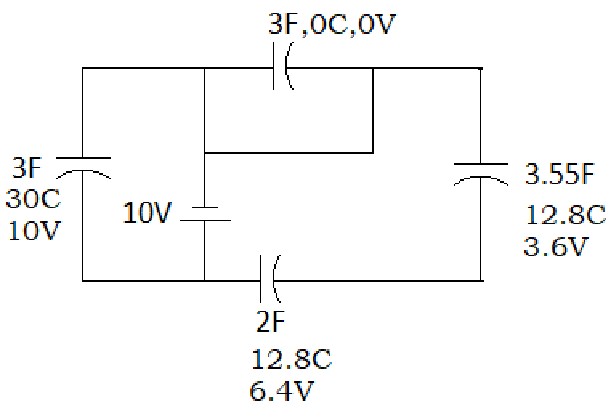
So the charge on this equivalent capacitor is q = CΔV = 42.8C.



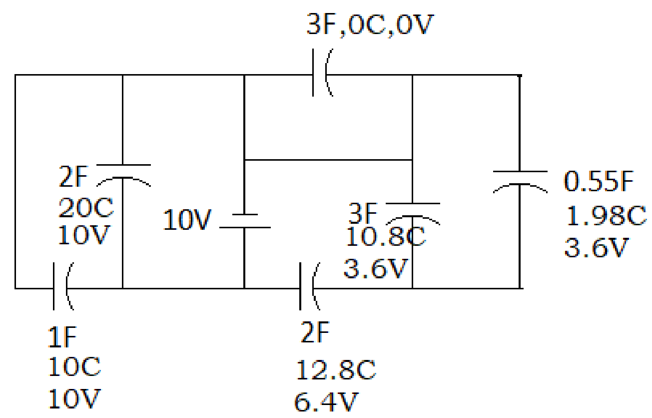
And now we gotta go back. The 4.28F capacitor expands in parallel into the following two. Since they're in parallel, the potential differences are all the same, 10V. And then the charges on them will be q = CΔV = (3,1.28)(10) = (30,12.8):



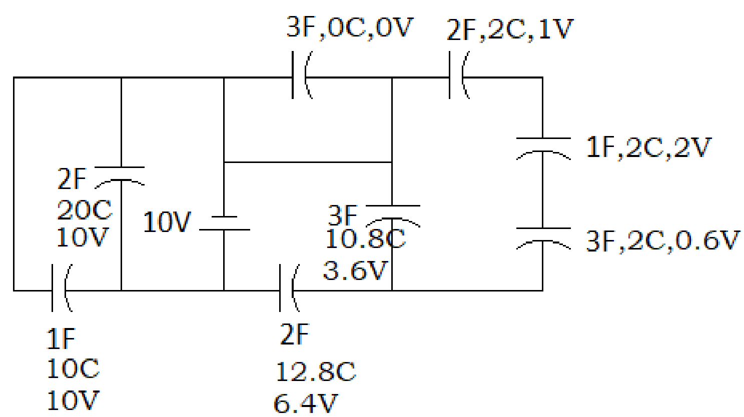
Going back to the previous one, one of the wires gets expanded into a wire + 3Fcapacitor. And we acknowledged that there would be no charge on it, and therefore no potential difference either. And the 1.28F capacitor gets expanded in series into a 2F and 3.55F cap. Series expansions preserve charge, rather than voltage so each of these will have a 12.8C charge. And their potential differences will be ΔV = q/C = 12.8/(2F,3.55F) = (6.4V, 3.6V)



Going back again, the 3F capacitor gets expanded in parallel into the 1F and 2F. Potential difference is preserved so these will both be 10V. And the charges will be q = CΔV = (1,2)(10) = (10,20). Also, the 3.55F capacitor gets expanded in parallel into the 3F and 0.55F. Again, the potential difference is preserved so these will both be 3.6V. And the charges will be q = CΔV = (3,0.55)(3.6) = (10.8,1.98)



And now back again, the 0.55F capacitor gets expanded in series into the 2F, 1F, and 3F caps. Charge will remain the same so each will carry 1.98C ≈2C. And their potential differences will be ΔV = q/C = (1.98)/(2F,1F,3F) = 1V, 2V, 0.6V.



(b) If the battery were disconnected and some device, like a motor, were connected to the circuit in its place, how much charge would flow through the device?
This would just be the charge on the equivalent capacitor, i.e., q = 42.8C.

(c) How much energy would be delivered to the device?
And this would be the total energy stored in the circuit, which is the same as the energy stored in the equivalent cap, which is:

$$PE = \frac{1}{2}C_{eq.}\Delta V^2 = \frac{1}{2}(4.28\text{F})(10\text{V})^2 = 214\text{J}$$